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The Sup Norm of a Polynomial with Perturbed Coefficients

J. S. BYRNES

Department of Mathematics, University of Massachusetts, Boston, Massachusetts 02125

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We show that if the coefficients of a polynomial $P(z)$ of degree n are perturbed by at most $t > 0$, then the order of magnitude of the sup norm of P on the unit circle is at most multiplied by $tn^{1/2}$. Furthermore, the polynomials with coefficients of modulus 1 of Kahane are used to show that this multiplication effect is achievable.

In this article we study the effect of the perturbation of the coefficients of a polynomial $P(z)$ on the sup norm of P on the unit circle. Thus, for $\varepsilon > 0$ and any positive integer n, we define an *ε-perturbation* of the $(n-1)$st degree polynomial $P(z) = \sum a_k z^k$ to be any $(n-1)$st degree polynomial $Q(z) = \Sigma\, a_k b_k z^k$, where (all sums in the paper are from $k = 0$ to $k = n-1$) the coefficients b_k satisfy

$$|b_k - 1| \leqslant \varepsilon. \tag{1}$$

Our task is to estimate the number $G = G(\varepsilon, n) = \text{Sup}\, \|Q\|/\|P\|$, where $\|\cdot\|$ indicates the sup norm on $|z| = 1$, P is any polynomial of degree $n-1$ with $\|P\| > 0$, and Q is in any ε-perturbation of P. We show that G is asymptotic to $\varepsilon n^{1/2}$ as $n \to \infty$. More precisely, we prove the following:

THEOREM. *Let G be as defined above. Then there is an absolute constant $C > 0$ such that* $1 + \varepsilon n^{1/2} - C\varepsilon n^{3/10}(\log n)^{1/2} < G \leqslant 1 + \varepsilon n^{1/2}$.

Proof. Applying (1), the Schwarz inequality, and Parseval's identity we have

$$|Q(z)| = \left| P(z) + \sum a_k(b_k - 1)\, z^k \right| \leqslant \|P\| + \sum |a_k|\, |b_k - 1|$$

$$\leqslant \|P\| + \varepsilon \sum |a_k| \leqslant \|P\| + \varepsilon n^{1/2} \left(\sum |a_n|^2\right)^{1/2} \leqslant (1 + \varepsilon n^{1/2})\, \|P\|,$$

which immediately yields the upper bound.

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For the lower bound we clearly want a polynomial P with small sup norm whose coefficients are roughly equal to each other in modulus. Thus the polynomials of Kahane [2], which disprove a well-known conjecture of Erdös [1], are ideally suited to our purpose. Specifically, Kahane has shown that there is an absolute constant $C > 0$ such that, for any positive integer n, there is a polynomial $K(z)$ of degree $n-1$, with coefficients of modulus 1, satisfying $\|K\| \leqslant n^{1/2} + Cn^{3/10} (\log n)^{1/2}$ (obviously $\|K\| \geqslant n^{1/2}$, as this is the L^2 norm of K on the unit circle). We may certainly assume, by a suitable normalization, that $\|K\| = K(1)$.

Taking $P(z)$ to be this normalized Kahane polynomial, let e^{it_k}, t_k real, denote the coefficients of P, and let $b_k = 1 + \varepsilon e^{-it_k}$. We then have $Q(z) = P(z) + \varepsilon \sum z^k$, so that $\|Q\| = Q(1) = P(1) + \varepsilon n$, or

$$\frac{\|Q\|}{\|P\|} = 1 + \varepsilon \frac{n}{\|K\|} > 1 + \varepsilon n^{1/2} - C\varepsilon n^{3/10} \quad (\log n)^{1/2},$$

and the Theorem is proven.

References

1. P. Erdös, Some unsolved problems, *Michigan Math. J.* **4** (1957), 291–300.
2. J. P. Kahane, Construction of a polynomial $P_N(z) = \sum_{n=1}^{N} \hat{P}_N(n) z^n$, $|\hat{P}_N(n)| = 1$, such that $\sup_{|z|=1} |P_N(z)| = N^{1/2} + O(N^{3/10+\epsilon})$, preprint.